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Noise delayed decay of unstable states

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The present paper is concerned with transient phenomena near the instability points of dynamic systems. The problem of decay of unstable states is investigated in the framework of the model of overdamped Brownian motion. It is shown that in many physical situations the additive noise can increase the decay time of unstable states, in contrast with accepted notions. The conditions for this effect are studied in detail. Physical examples are considered. [S1063-651X(98)04203-2]

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I. INTRODUCTION

Time characteristics of the decay of unstable states that appear in various dynamical systems have been studied in many works (see Refs. [1–19], and references therein). Following from these works, in this paper, the process of the decay is considered within the model of overdamped Brownian motion in potential fields. The coordinate x of the Brownian particle is supposed to obey the following Langevin equation with the additive noise source:

$$\frac{dx}{dt}=-\frac{dU(x)}{\eta dx}+\xi(t), \tag{1}$$

where $U(x)$ is potential, $\xi(t)$ is the Gaussian white noise, $\langle\xi(t)\rangle=0$, $\langle\xi(t)\xi(t+\tau)\rangle=D\delta(\tau)$, D is the noise intensity, and η is the viscosity, which further can be taken equal to unity. In this paper the potential profiles $U(x)$, which have no local minima, are considered. Such potential profiles describe the unstable states of dynamic systems. The decay time of the states is defined as the mean first passage time (MFPT) of the Brownian particle across given boundaries. This approach to define the decay time of the unstable states is widely used, because the mathematical method for obtaining the moments of FPT distribution is well developed [20–22]. The analysis carried in Refs. [11–17] by use of the MFPT method shows that the decay time always decreases with noise intensity. In other words, the noise was shown to accelerate the decay of any unstable state.

At the same time in the work of Hirsch, Huberman, and Scalapino [18], the dependence on noise intensity of the decay time of an unstable state was revealed (analytically and numerically) to have something of a resonant character: with the growth of the noise intensity D, the decay time also grows in the beginning, then reaches a maximum, and decreases till zero under $D\to\infty$. However, the increase in the decay time obtained in Ref. [18] was not large (less than 1%). Moreover, in the work of Landa and Stratonovitch [19], some mistakes in the theoretical part of Ref. [18] were pointed out and, as in the above mentioned works, an increase in the decay time with noise intensity was not detected. Thus the problem has remained unsolved.

On the other hand, the effect of the noise enhanced stability of the unstable states was observed recently for systems driven both by stochastic and periodic forces [23–26]. However, these works explain the effect by the influence only of the periodic force.

The present work shows that the decay time of the unstable state under some conditions can be increased considerably by the external noise. In other words, the external additive noise, in contrast to the usual notions [4–17,19], can delay the decay of the unstable state.

The results presented here are a further development and generalization of the approach proposed in Ref. [28]. In that work the simplest model of a piecewise linear potential profile, which leads to the effect of the delay, was considered. In addition to Ref. [28], the influence of the potential profile shape and initial conditions on the effect is studied in the present work. For this, the decay times of unstable states described by the more complex potential profiles (piecewise linear and smooth) are considered. Section II is concerned with an analysis of the decay time of the unstable state described by a piecewise linear potential profile consisting of three linear parts. The general conditions under which the noise enhances the stability of the unstable state are formulated. In Secs. III and IV, some physical examples are discussed. In these sections we consider smooth potential profiles corresponding to the real systems. Section III is devoted to the influence of the external noise on intermittent chaotic systems. In Sec. IV we study the influence of noise on the bifurcation transitions.

II. DECAY TIME OF THE UNSTABLE STATE DESCRIBED BY THE PIECEWISE LINEAR POTENTIAL PROFILE

Let us consider the following potential profile consisting of three linear parts $x\in[-x_m,x_m]$ (Fig. 1):

$$U(x)=\begin{cases} -k_m(x-\ell_c)-k_c\ell_c, & -x_m=-\ell_m-\ell_c<x<-\ell_c \\ -k_c x, & -\ell_c<x<\ell_c \\ -k_m(x+\ell_c)+k_c\ell_c, & \ell_c<x<x_m=\ell_c+\ell_m, \end{cases} \tag{2}$$

where $k_c, k_m > 0$ are the slopes of the central and two marginal linear intervals of the profile, and ℓ_c and ℓ_m are the lengths of these intervals. If $k_c, k_m > 0$, this profile has no horizontal pieces, i.e., it describes an unstable state.

In accordance with Eq. (1), if the noise is absent, the Brownian particle initially located at some point $x_0 \in [-x_m, x_m]$ will slope down to the boundary x_m, and will reach it in time $T_m(x_0)$. If there is noise $\xi(t)$ in the system, the particle is subjected to random pushes; then the possibility of leaving the interval through the upper boundary appears, and the time $\tilde{T}(x_0)$ of the escape from the interval becomes random. Let the decay time of the unstable state be the MFPT $\langle \tilde{T}(x_0)\rangle \equiv T(x_0,D)$ of the Brownian particle initially located in the point x_0 across the boundaries of the interval $[-x_m, x_m]$.

To obtain the MFPT, one can use the well-known formula [20,22]

$$T(x_0,D)=\frac{2}{D}\left[\int_{x_0}^{x_m} e^{u(\zeta)}\int_{-x_m}^{\zeta} e^{-u(\phi)}d\phi\, d\zeta - \frac{\int_{-x_m}^{x_m} e^{u(\zeta)}\int_{-x_m}^{\zeta} e^{-u(\phi)}d\phi\, d\zeta}{\int_{-x_m}^{x_m} e^{u(\zeta)}d\zeta}\int_{x_0}^{x_m} e^{u(\zeta)}d\zeta\right], \tag{3}$$

where $u(x)=2U(x)/D$ is dimensionless potential profile. Also one can use the method proposed in Ref. [27], which is based on first obtaining the Laplace transform of the solution of the Fokker-Planck equation (FPE) for the probability density $W(x,t)$ corresponding to Eq. (1):

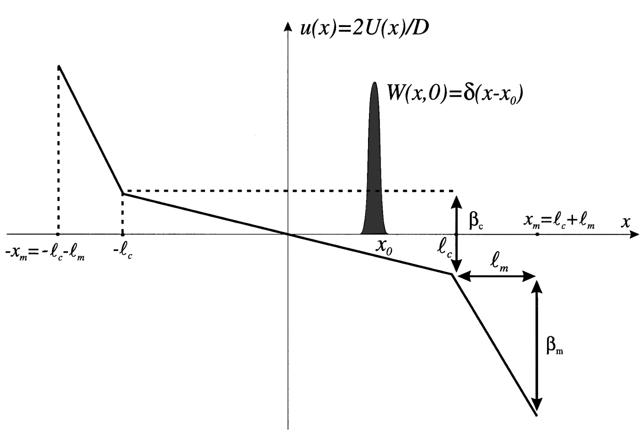

FIG. 1. The piecewise linear potential profile (2).

$$\frac{\partial W(x,t)}{\partial t}=\left[\frac{\partial}{\partial x}\frac{dU(x)}{dx}+\frac{D}{2}\frac{\partial^2}{\partial x^2}\right]W(x,t). \tag{4}$$

Using one of the above mentioned methods, one can obtain an exact expression for the MFPT $T(x_0,D)$, which under low noise intensity, when $\exp(-\beta_c)\ll 1$ and $\exp(-\beta_m)\ll 1$, where $\beta_c=4k_c\ell_c/D$, $\beta_m=2k_m\ell_m$, and $\ell_m=x_m-\ell_c$ (see Fig. 1), will have the following view:

$$\Theta(X_0,D)\equiv\frac{T(X_0,D)}{T_m}=\begin{Bmatrix} \Theta_l(X_0,D), & -1<X_0<-P \\ \Theta_c(X_0,D), & -P<X_0<P \\ \Theta_r(X_0,D), & P<X_0<1 \end{Bmatrix} -0(e^{-\beta_m})-0(e^{-\beta_c}), \tag{5}$$

where

$$\Theta_l(X_0,D)=1-X_0(L+1)-L+2K^{-1}L+\frac{2L}{\beta_c}(2-K-K^{-1}) -\left(2+2K^{-1}L+\frac{2L}{\beta_c}(2-K-K^{-1})\right)\times e^{-\beta_m(L+1)(1+X_0)},$$

$$\Theta_c(X_0,D)=1+K^{-1}L-K^{-1}(L+1)X_0+\frac{2L}{\beta_c}(1-K)+\frac{2L}{\beta_c}(1-K^{-1})e^{-\beta_c[1+X_0(1+L^{-1})]/2},$$

$$\Theta_r(X_0,D)=1+L-(L+1)X_0+\frac{2L}{\beta_c}(1-K)\times e^{-\beta_m[X_0(L+1)-L]}.$$

The new values introduced here are as follows: $L=\ell_c/\ell_m$ is the relative width of the central piece, $K=k_c/k_m$ is the relative slope of the central piece, $X_0=x_0/x_m$, $P=L/(L+1)$, and time $T_m=\ell_m/k_m$. The dependence of the decay time of the unstable state $T(X_0,D)$ [Eq. (5)] on the noise intensity D is contained in the values $\beta_m,\beta_c\sim 1/D$ that characterize the dimensionless height of the two linear parts of the potential profile (2). It follows from Eq. (5) that if $D=0$ ($\beta_m=\beta_c=\infty$), then $T(X_0=P,0)=T_m$. This means that T_m is the time of the descent of the particle from the point $x_0=\ell_c$ to the boundary $x_m=\ell_c+\ell_m$ on the right, marginal piece of potential profile (2), of which the longitude is ℓ_m and the slope is k_m.

First of all, it is necessary to note that the decay time of the unstable state (5) can be an increasing function of the noise intensity. As one can easily see from Eq. (5), if

$$K<1, \quad -P<X_0<1,$$

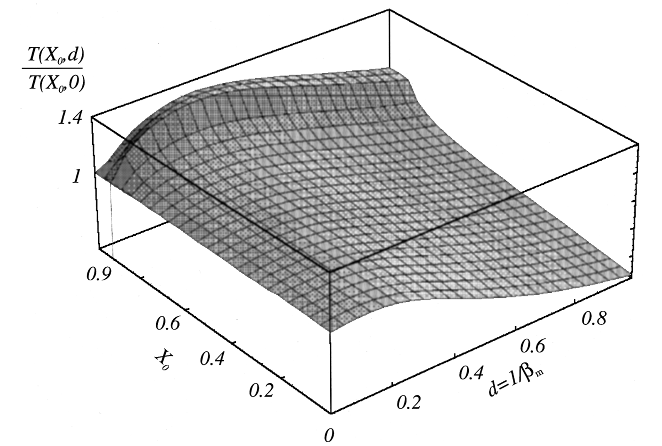

FIG. 2. The dependence of the relative MFPT for the piecewise potential profile (2) on the dimensionless noise intensity $d=1/\beta_m$ under $K=0.6$, $L=10$, and $P=0.9$ ($\ell_c=0.9x_m$).

the derivative $\partial T(X_0,D)/\partial D>0$. On the other hand, it follows from the exact expression for $T(X_0,D)$ (which we do not adduce here, because it is huge) that under large D, when $\exp(-\beta_c)$, $\exp(-\beta_m)\sim 1$, the MFPT decreases as $1/D$. Consequently under some noise intensity $D=D^*$, the MFPT has a maximum $T(X_0,D^*)=T^*$ (see Fig. 2).

Thus, the results obtained [Eq. (5)] allow one to see the effect of the noise enhanced stability of the unstable state. This effect was observed in Ref. [28] for a more simple kind of potential profile, and under fixed initial conditions.

In order to understand better how this effect appears, let us consider the system which is governed by Eq. (1), where, instead of the random force $\xi(t)$, there is a δ function with a random amplitude a:

$$\dot{x}=-\frac{dU(x)}{dx}+a\,\delta(t).$$

In such a system the Brownian particle undergoes the only random push at the moment $t=0$. It can easily be shown that the action of δ impulse leads to the shift of the initial position of the particle on the distance a. Let the distribution of the amplitude $W(a)$ be the even function. For example, one may take

$$W(a)=\tfrac{1}{2}\,\delta(a+a_0)+\tfrac{1}{2}\,\delta(a-a_0).$$

Initially, let this particle be at the point $X_0=P$ ($x_0=\ell_c$), i.e., at the boundary of the two low linear pieces of potential profile (2) (see Fig. 1); the amplitude of the impulse does not exceed the lengths of these pieces: $a_0<\ell_m$, $a_0<\ell_c$. In this case, if $a=-a_0$ (at the moment $t=0$ the particle is shifted on the distance $-a_0$), then the escape time from the interval is equal:

$$\tau(-a_0)=\frac{a_0}{k_c}+\frac{\ell_m}{k_m}.$$

If $a=a_0$, then

$$\tau(a_0)=\frac{\ell_m-a_0}{k_m}.$$

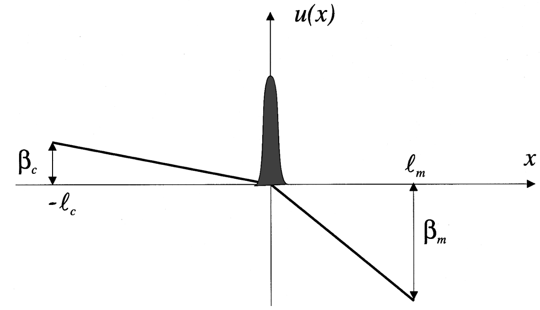

FIG. 3. The piecewise linear potential profile considered in Ref. [28].

Consequently, the average escape time from the interval is

$$\langle\tau(a)\rangle=\int_{-\infty}^{\infty}\tau(a)W(a)da=T_m+a_0\left(\frac{1}{k_c}-\frac{1}{k_m}\right),\qquad(6)$$

where $T_m=\tau(0)=\ell_m/k_m$ [as in Eq. (5)]. It follows from Eq. (6) that, under $k_c=k_m$ ($K=1$), the random push does not effect the average escape time. Under $k_c>k_m$ ($K>1$), the average escape time decreases with the increase in the amplitude a_0, and under $k_c<k_m$ ($K<1$) this time increases with the increase in a_0, i.e., the greater the amplitude of the δ impulse, the longer the time the particle stays within the interval.

As it was shown, for example, in Ref. [29], the completely random process can be represented as a superposition of the δ impulses. Therefore, it is evident that the appearance of the effect of a noise induced increase of the unstable state decay time in the system described by Eq. (1), as in the above mentioned example, is due to the nonlinearity of the potential profile. If the particle is on the upper and less tilted ($k_c<k_m$) piece of the profile, then it is delayed there for a long time. This leads to an increase of the average escape time from the interval. In accordance with this, the less the relative slope $K=k_c/k_m$, and the more the relative length $L=\ell_c/\ell_m$ of this piece, the greater the maximum of the MFPT. This qualitative conclusion is confirmed by Eq. (5).

Let us now discuss the influence of the initial conditions. The plot $T(X_0,D)/T(X_0,0)$ for the different values X_0, drawn in accordance with Eq. (3), is presented in Fig. 2. It follows from Eqs. (3) and (5) that the maximal relation $T(X_0,D)/T(X_0,0)$ is reached under the some noise intensity $D=D^*$, if the diffusion is started from the point $x_0=\ell_c$ ($X_0=P$), which is located on the boundary of the two lowest pieces of potential profile (2).

It must be mentioned that one can come to the same conclusion if one considers diffusion with arbitrary initial conditions in a model potential profile consisting only of two linear pieces (Fig. 3). This potential was considered in Ref. [28]. It can be obtained from the potential profile (2), if we remove the upper piece at $x<-\ell_c$. If we take the arbitrary initial conditions x_0 in the problem with such a ''shortened'' potential, we obtain the expression for the MFPT $\hat{T}(X_0,D)$ which coincides with Eq. (5) under the small noise intensity D [when $\exp(-\beta_c)$, $\exp(-\beta_m)\ll 1$]. Taking into account the fact that the effect of the increase of the MFPT by noise is observed under low noise intensity, we may conclude that the piece of the potential profile under $x<-\ell_c$ does not influence the main characteristics of the effect. Thus, if one changes the slope and the length of this piece arbitrarily, it will not influence the characteristics of the effect.

The value of the noise intensity D^* corresponding to the maximum value of MFPT approximately may be estimated as follows:

$$D^* \approx U_c + U_m ,$$

where $U_c = k_c \ell_c$, $U_m = k_m \ell_m$. To obtain the exact expression for D^* is extremely difficult, because of the mathematics. Thus one can distinguish the following main features of the effect of the noise enhanced stability of unstable states.

(1) In order for the effect to arise, at least two pieces with different slopes k_1 and k_2 $(k_1 k_2 > 0)$ must be distinguished in a potential profile. In the case of the potential profile (2), they are the middle and the lower linear pieces.

(2) The effect arises if the slope k_{lower} of the lower piece is more then the slope k_{upper} of the neighboring upper piece. For the above case, $k_{\mathrm{lower}} = k_m$ and $k_{\mathrm{upper}} = k_c$.

(3) The effect manifests itself maximally if diffusion starts from the point located on the boundary of these two pieces.

(4) This effect is stronger [i.e., the maximal value of MFPT $T(x_0, D)$ is the greater] the more difference there is in the slopes of the pieces (i.e., the less $K = k_c / k_m$), and the more the relative value $L = \ell_c / \ell_m$ of the upper piece.

(5) The shape of the part of the potential profile, which is located behind the upper piece, influences the characteristics of the effect inconsiderably. In the case of the potential profile (2) this part is located at the interval $-x_m < x < -\ell_c$.

III. INFLUENCE OF THE EXTERNAL NOISE ON THE INTERMITTENT SYSTEMS

The main conclusions obtained by analysis of the diffusion in the model piecewise linear potential profiles are useful, when the diffusion in smooth potential profiles typical of the real physical systems is studied. Let us consider, for instance, the influence of the external noise on an intermittent chaotic system. It is well known (see, e.g., Refs. [12,17–19], and references therein) that the intermittent system is chaotically switched under the constant external parameters between two regimes: laminar and chaotic. One may say that under intermittency the laminar phase of the system becomes unstable.

The main characteristic of the intermittency is the average path length (APL) or, in other words, average duration of the laminar regime. In this section we consider the influence of the external noise on the APL for the type 1 intermittency, which arises, for example, in a logistic map

$$x_{n+1} = F(x_n, R) = R x_n (1 - x_n) \tag{7}$$

under $R > R_c = 1 + \sqrt{8}$. This problem can be expressed in terms of Brownian motion, and the behavior of the dynamical system can be described by the Langevin equation (1) with the potential describing the unstable state [12,17–19]. The view of the potential profile $U(x)$ is defined by the function $F(x_n, R)$.

The following potential profile describing the unstable state corresponds to the type 1 intermittency:

$$U_{\mathrm{int}}(x) = -Ax - Bx^z, \tag{8}$$

where $z = 2n+1$, $n = 1,2,3, \ldots$, $A > 0$, and $B > 0$. For the case of the logistic map (7), $A = R_c - R$, $B = 68.5$, and $z = 3$.

The diffusion of the Brownian particle in the potential field (8) in the interval $x \in [-x_m, x_m]$ (the value x_m should satisfy $A + zBx_m^{z-1} \ll 1$) corresponds to the laminar phase. The escape of the Brownian particle from the interval corresponds to the transition of the system into the chaotic regime. Thus the MFPT across the boundaries $[-x_m, x_m]$ characterizes the APL [18].

In the problem of the intermittency, the initial point x_0 is the point the variable x reaches after the transition from the chaotic behavior to the deterministic one. The variable x_0 is random, and characterized by a distribution $\omega(x_0)$, which in the general case is defined by the particular model of the chaotic system. Following [18], as the first approximation to the real distribution, we will consider the uniform probability density

$$\omega(x_0) = \frac{1}{2x_m}, \quad x_0 \in [-x_m, x_m]; \tag{9}$$

then the APL $\overline{T(D)}$ will be

$$\overline{T(D)} = \langle T(x_0, D) \rangle = \frac{1}{2x_m} \int_{-x_m}^{x_m} T(x_0)\, dx_0 . \tag{10}$$

It is necessary to take into account that if

$$\exp(2 |U(x_0) - U(-x_m)| / D) \gg 1, \tag{11}$$

i.e., if the noise intensity D is small, then one can neglect the second term in Eq. (3). However, if we consider an initial distribution similar to Eq. (9) (where x_0 distributed over the whole interval $[-x_m, x_m]$), then we can not neglect by the second term in Eq. (3), as was done in Ref. [18].

Unfortunately, the solution of the FPE (4) for the potential functions (8) is unknown. Therefore the only way to define the exact value of the MFPT $T(x_0, D)$ for potential (8) is by using the integration expression (3), which can be evaluated in this case only numerically. In this situation the preliminary analysis of this problem with the help of the model potentials has special interest.

Let us consider the potential profile (8). In such a smooth potential one can also distinguish the three pieces with different characteristic slopes. One piece is located near $x \approx 0$, where the influence of the second term $\sim x^z$ in Eq. (8) is not considerable. Therefore the characteristic slope of this piece is equal to $-A$. With the growth of $|x|$ the influence of the second term increases and becomes dominant. Thus two margin pieces with larger characteristic slopes appear. Consequently, the effect of the noise induced increase of MFPT should take place in this system.

In order to obtain information on the main characteristics of this effect, we can approximate the smooth potential profile (8) to the piecewise linear one (2). Because the first term in potential (8) is linear, we need to define how to approximate the second term $-Bx^z$ only. From the viewpoint of the considered effect, the main feature of this term is that, under

$|x| \ll \ell_c$, we can neglect it, and under $|x| > \ell_c$ it becomes significant. Let ℓ_c be the point where both terms are equal: $A\ell_c = B\ell_c^z$. Then we can propose the following way to replace the function Bx^z by the three linear pieces:

$$Bx^z \to G(x) \equiv \begin{cases} \dfrac{Bx_m^z}{x_m - \ell_c}(x + \ell_c), & -\ell_c < x < -x_m \\ 0, & -\ell_c < x < \ell_c \\ \dfrac{Bx_m^z}{x_m - \ell_c}(x - \ell_c), & \ell_c < x < x_m. \end{cases} \tag{12}$$

The slope of the linear pieces under $\ell_c < |x| < x_m$ is chosen to satisfy the condition $G(\pm x_m) = \pm Bx_m^z$ and the condition of continuity of $G(x)$ at the points $|x| = \ell_c$. This approximation assumes that, under $\ell_c = (A/B)^{1/(z-1)} > x_m$, one can neglect the second term.

Thus, taking into account the first term, we obtain the following piecewise linear potential profile, which models the initial smooth one (8):

$$U(x) = -Ax - G(x). \tag{13}$$

For this potential profile the dimensionless parameters—the relative slope K and the relative length L of the central piece—are as follows:

$$K(m,z) = \frac{m^{1/(z-1)} - 1}{m^{z/(z-1)} + m^{1/(z-1)} - 1}, \tag{14}$$

$$L(m,z) = \frac{1}{m^{1/(z-1)} - 1}, \tag{15}$$

where $m = Bx_m^{z-1}/A$ is the dimensionless parameter characterizing the shape of the initial potential profile (8), which can be represented as follows:

$$u_{\rm int}(X) = \frac{2U_{\rm int}(X)}{D} = -(X/m + X^z)/d, \tag{16}$$

where $X = x/x_m$ is the dimensionless coordinate, and $d = D/2Bx_m^z$ is the dimensionless noise intensity.

Now, when we know how the parameters K and L of the piecewise linear potential profile depend on the dimensionless values m and z defined by the shape of the approximated smooth potential profiles (8) and (16), we can analyze the dependence $T(x_0, D)$ qualitatively for the real potential, and investigate how the variation of the parameters m and z influences the maximal value of the MFPT:

$$\max_D \{T(X_0, D)\} \equiv T^*(X_0).$$

As follows from the conclusions of Sec. II, the less the relative slope of the central piece $K(m,z)$ is, i.e., in accordance with Eq. (14), the greater the parameter m is, the greater the maximum of MFPT $T^*(x_0)$ is. However, on the other hand, it follows from Eq. (15), that with an increase of m, the relative width of the central piece $L(m,z)$ is decreased. As discussed in Sec. II, this leads to a decrease of T^*. Thus we may conclude that the variation of the parameter m cannot have a considerable affect on the maximal value $T^*(x_0)$ of the MFPT.

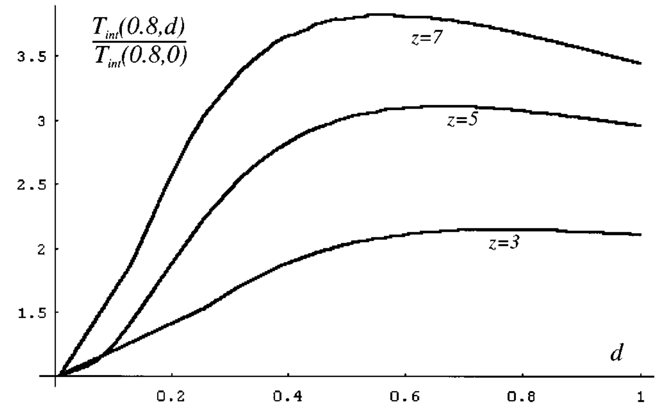

FIG. 4. The dependence of the relative MFPT across the boundaries $-x_m$ and x_m for the potential profile (8) on the dimensionless noise intensity $d = D/2Bx_m^z$ under $m = Bx_m^{z-1}/A = 5$ and the initial condition $x_0 = 0.8x_m$.

Let us consider the influence of parameter z on $T^*(x_0)$. It follows from Eqs. (14) and (15), that the relative slope of the central piece K is decreased, and its relative width L is increased with the growth of z. In accordance with the conclusions of Sec. II, this should lead to an increase of $T^*(x_0)$, i.e., the variation of the parameter z affects the maximal MFPT T^*: the more z is, the more T^* is.

All these conclusions of the qualitative analysis are confirmed by numerical calculations of the integral expression (3) for the approximated smooth potential profile (16). In Fig. 4 the dependencies of the dimensionless value $T_{\rm int}(X_0,d)/T_{\rm int}(X_0,0)$ on the dimensionless noise intensity $d = D/2Bx_m^z$ are plotted under $X_0 = 0.8$ ($x_0 = 0.8x_m$) and $m = 5$ for different values of z. One can see that the maximal value of the relative MFPT is increased with the growth of z.

Note that the influence of initial conditions on the MFPT in the case of a smooth potential profile is not absolutely the same as in the case of a piecewise linear one. The maximal value $T_{\rm int}^*(X_0)$ is not observed under $X_0 = P$, which is the boundary point of the two pieces, but under $P < X_0 < 1$: the more X_0 is, the more $T_{\rm int}^*(X_0)$ is. This is explained by the fact that, under $X > \ell_c/x_m = P$, the potential profile (16) can be replaced by the linear one only conditionally. It is evident that in every point $X = X_0$ of the real potential profile (16), and especially under $X_0 > \ell_c/x_m$, the relative slope on the left from X_0 is less than on the right from this point, and the greater X_0 is, the greater the difference. Therefore, $T_{\rm int}^*(X_0)$ increases with the increase of X_0.

Thus the effect of the increase of the MFPT $T_{\rm int}(x_0,D)$ by the external noise takes place for the potential profiles (8) and (16), and can be considerable. It is defined mainly by the initial conditions and by the value z characterizing the steepness of the potential profile, while the influence of the parameter m characterizing the relative contribution of the linear term in Eq. (16) is not considerable.

Let us consider now the dependence of the APL $\overline{T(D)}$ on the noise intensity. In accordance with Eq. (10), in order to obtain the APL $\overline{T(D)}$ we must average $T(x_0,D)$ on initial

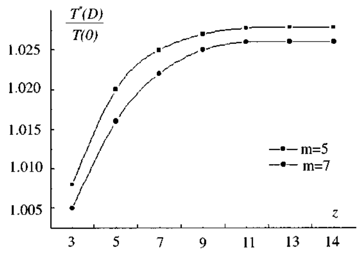

FIG. 5. The dependence of the maximal APL (10) on the parameter z of potential profile (8) under $m=5$ and 7.

conditions x_0. If we average the expression for $T(x_0,D)$ corresponding to the model potential profile (13), we obtain the APL $\overline{T(D)}$, the derivative of which is $d\overline{T(D)}/dD<0$ for any D, K, and L. However, if we calculate the APL corresponding to the smooth potential profile (8) numerically, we find that there is a maximum of $\overline{T}_{\rm int}(D)$ under some $D^*>0$. It is evident that the difference in the behavior of the APL's with the increase in the noise intensity for the model potential profile (13) and for the smooth one (8) is explained by the different dependencies of the MFPT's $T(x_0,D)$ and $T_{\rm int}(x_0,D)$ considered above for these potentials on the initial conditions x_0.

The dependence of the maximal value $\overline{T^*_{\rm int}}$ of APL (10) $[\overline{T^*_{\rm int}}\equiv\max\{\overline{T_{\rm int}}(D)\}]$ on the parameters of the potential profile is similar to that of MFPT $T^*_{\rm int}(x_0)$ considered above: the increase of the parameter z leads to the increase of $\overline{T^*_{\rm int}}$, while the influence of the parameter m is less important. However, under $z>7$ the value $\overline{T^*_{\rm int}}$ becomes a constant not dependent on z, and comes out on a level which does not exceed 3–4 % in comparison with $\overline{T}_{\rm int}(D=0)$. The dependence $\overline{T^*_{\rm int}}(z)$ for some values of m is presented in Fig. 5.

Thus we may conclude that the effect of the increase of the APL by the external noise exists, and can be essential for the some initial distributions $\omega(x_0)$. On the other hand, for the considered uniform distribution (9), which is typical for the logistic map, this effect is inconsiderable. It follows from the above analysis, that the effect will be stronger, if we take the initial distributions $\omega(x_0)$, which have a maximum at $0<x_0<x_m$. The value $\overline{T^*_{\rm int}}$ will be greater, the closer this maximum to the boundary x_m.

IV. INFLUENCE OF THE EXTERNAL NOISE ON THE BIFURCATION TIMES

In this section we consider states of the dynamic systems which lost their stability due to the quick change of the appropriate bifurcation parameters. There are two types of potential profiles, which correspond to such states. The first one is the same as Eq. (8), i.e.,

$$U_1(x)=U_{\rm int}(x)=-Ax-Bx^z, \tag{17}$$

$z=2n+1$, $n=1,2,3\ldots$, $A\geqslant0$, $B>0$, $U_1(x)\to+\infty$ under $x\to-\infty$, and $U_1(x)\to-\infty$ under $x\to+\infty$. The unstable state of the system in this case is supposed to be at the interval $x\in[-\infty,x_m]$ (see, e.g., Refs. [10,15,16]). This kind of potential appears, for example, when the phenomena of optical bistability is studied. The second type of potential profile of an unstable state is symmetric, and appears when we consider the soft regime of the excitation of oscillators (e.g., the so-called laser model [1,6]) or phase transitions of the second order [5,13],

$$U_2(x)=-Ax^z, \tag{18}$$

where $z=2n$ $[U_2(x)\to-\infty$ when $x\to\pm\infty]$. The unstable state in this case is located at the interval $x\in[-x_m,x_m]$.

The decay time $T_1(x_0,D)$ of an unstable state of the first kind is the MFPT across the boundary x_m:

$$T_1(x_0,D)=\frac{2}{D}\int_{x_0}^{x_m}e^{u_1(\zeta)}\int_{-\infty}^{\zeta}e^{-u_1(\phi)}d\phi d\zeta. \tag{19}$$

As mentioned before, it coincides with the above considered decay time (3) in the interval $x\in[-x_m,x_m]$ under condition (11). Thus the results of the analysis presented in Sec. III are applicable under small noise intensity D for the first kind of unstable state considered in this section. Therefore, we can conclude that the decay time of this unstable state can be increased by noise. The effect is greater the closer x_0 is to x_m and the greater z is.

In addition, using the method proposed by Malakhov in Ref. [30] we can find the following expansion of the decay time (19) in terms of the noise intensity:

$$\begin{aligned}T_1(x_0,D)=&\int_{x_0}^{L}y(v)dv+\frac{1}{2}\frac{D}{2}[G_1(x_0)-G_1(L)]\\&+\left(\frac{D}{2}\right)^2\int_{x_0}^{L}y(yy')'dv+\frac{1}{2}\left(\frac{D}{2}\right)^3[G_3(x_0)\\&-G_3(L)]+\left(\frac{D}{2}\right)^4\int_{x_0}^{L}y[y(y(yy')')']'dv+\cdots,\end{aligned} \tag{20}$$

where

$$\begin{aligned}G_1(v)&=y^2,\\G_3(v)&=y^2[-(y')^2+2(yy')'],\\y&=y(v)=-1/U_1'(v).\end{aligned} \tag{21}$$

Series (20) is valid for a potential profile of the first kind [Eq. (17)] under $A>0$ only. The first term in Eq. (20) is the time of escape from the interval, when noise is absent. It is easy to see from Eq. (21) that the second term is positive for any $z>1$, if $-x_m<x_0<x_m$. Thus it follows from Eq. (20), that if these conditions are fulfilled, the noise increases the decay time of the unstable state described by the potential profile (17).

Let us now consider a potential profile of the second kind [Eq. (18)]. If the system is initially in a state corresponding exactly to $x_0=0$, then evidently the effect of the increase of the MFPT $T(0,D)$ by noise cannot appear, because, in this case, when the noise is absent, $T(0,0)=\infty$. However, if $x_0\neq0$, then $T(x_0,0)$ is a finite value, and the effect should take place, since, in the potential profile (18), one can distinguish

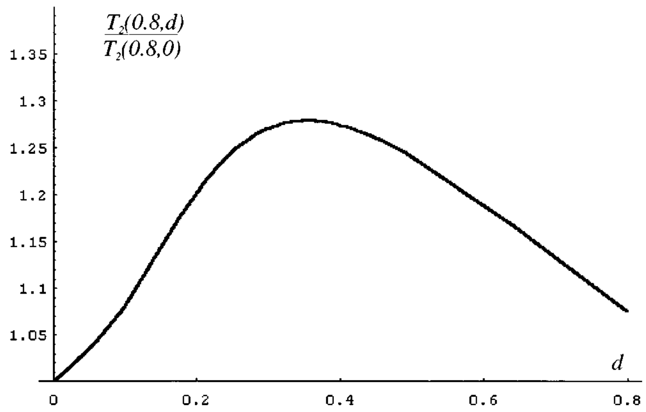

FIG. 6. The dependence of the MFPT across the boundaries $-x_m$ and x_m on the dimensionless noise intensity for the potential $U_2=-Ax^2$ under $x_0=0.8x_m$.

regions with different characteristic slopes: the first one is near $x\approx 0$ with a smaller slope, and the other one is near x_m with a larger slope. In accordance with the conclusions of Sec. II, the effect of the increase of the MFPT $T(x_0,D)$ by noise takes place in such a system.

In Fig. 6 the plot of the relative MFPT $T_2(X_0,d)/T_2(X_0,d)$ versus the dimensionless noise intensity $d=D/2Ax^z$ is presented. This plot is calculated using Eq. (3) under $X_0=0.8$ ($x_0=0.8x_m$) for the potential profile (18) with $z=2$. The decay time of this unstable state is seen to be increased by the external noise by more than 25%.

The unstable state described by the potential profile (18) under $z=2$ has special interest, since in this case the dynamic system governed by Eq. (1) is linear, and a further analysis of the effect in this system should not meet any considerable difficulties. As mentioned in Sec. I, the decay time of these states was studied in the literature earlier [11–17]. However, the effect of the noise enhanced stability was not detected. This was because the influence of initial conditions on the decay times was not investigated in much detail. In the above works the scaling methods were used to obtain the time characteristics of the decay. These methods suppose that the noise influences the decay mainly in the region of the potential profile $U(x)$ where the regular force $F(x)=-U'(x)$ is minimal; e.g., for potential (18) it is near $x=0$. Beyond this region the diffusion of the Brownian particle is supposed to have a deterministic character. That is why the MFPT's for initial conditions located far from the region where $U'(x)\approx 0$ were investigated only superficially. It follows from the above analysis that, if we take the initial conditions just in these regions, where the regular force is strong, the noise can increase the decay time.

V. CONCLUSION

In this paper the effect of the noise enhanced stability of unstable states was shown to appear in various physical systems, described by different kinds of potential profiles. When this effect takes place, the dependence of the decay time on the noise intensity has something of a resonant character, and there is a noise intensity $D=D^*$ for which the decay time is maximal. Another example of the physical system where this effect can appear was considered in Ref. [31].

The general conditions under which the effect appears as formulated in Sec. II are useful for a preliminary analysis of the effect in various dynamic systems. The effect may appear, if the diffusion of the Brownian particle starts from the region where the regular force is strong in comparison with the random one. Therefore, the influence of the noise on the escape time from these regions can be essential as well. The periodic force introduced in the systems considered in Refs. [23–26] cannot be the only cause of the noise enhanced stability of the unstable states, because in the present paper, this effect is shown to appear without the periodic drive.

The above analysis does not take into account that the Brownian particle can return to the interval under consideration after it has crossed the boundary once. This restriction is due to the use of the first passage time method. Further analysis, free from this limitation, would be very useful. Finally, it must be pointed out that in spite of the numerical tests presented here, providing strong confirmation of the analytic results, experimental verification of the effect would be very desirable.

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